

BOUNDEDNESS OF THREEFOLDS OF FANO TYPE WITH MORI FIBRATION STRUCTURES

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ABSTRACT. We show boundedness of 3-folds of ϵ -Fano type with Mori fibration structures. The proof is based on the birational boundedness result in our previous work [7] combining with arguments in Kawamata [9] and Kollár–Miyaoka–Mori–Takagi [14].

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1. INTRODUCTION

Throughout this paper, we work over the field of complex numbers \mathbb{C} . See Subsection 2.1 for notation and conventions.

A normal projective variety X is of ϵ -Fano type if there exists an effective \mathbb{Q} -divisor B such that (X, B) is an ϵ -klt log Fano pair.

We are mainly interested in the boundedness of varieties of ϵ -Fano type. Our motivation is the following conjecture due to A. Borisov, L. Borisov, and V. Alexeev.

Conjecture 1.1 (BAB Conjecture). *Fix an integer $n > 0$, $0 < \epsilon < 1$. Then the set of all n -dimensional varieties of ϵ -Fano type is bounded.*

BAB Conjecture is one of the most important conjectures in birational geometry and it is related to the termination of flips. Besides, since varieties of Fano type form a fundamental class in birational geometry according to Minimal Model Program, it is very interesting to understand the basic properties of this class, such as boundedness.

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BAB Conjecture in dimension two was proved by Alexeev [1] with a simplified argument by Alexeev–Mori [3]. In higher dimension, BAB Conjecture still remains open. There are only some partial boundedness results (cf. [5, 13, 9, 14, 2]).

We recall the following theorem proved in [6] by using Minimal Model Program.

Theorem 1.2 (cf. [6, Proof of Theorem 2.3]). *Fix an integer $n > 0$ and $0 < \epsilon < 1$. Every n -dimensional variety X of ϵ -Fano type is birational to an n -dimensional variety X' of ϵ -Fano type with a Mori fibration structure.*

According to this theorem, it is important and interesting to investigate varieties of ϵ -Fano type with Mori fibration structures. In fact, in the proof of BAB Conjecture in dimension two (cf. [1, 3]), the first step is to classify (and bound) all surfaces of ϵ -Fano type with Mori fibration structures, which are just projective plane or Hirzebruch surfaces \mathbb{F}_n with $n < 2/\epsilon$. Therefore, we are interested in the boundedness of 3-folds of ϵ -Fano type with Mori fibration structures, as the first step towards BAB Conjecture in dimension three.

The following is our main theorem.

Theorem 1.3. *Fix $0 < \epsilon < 1$. The set of all 3-folds of ϵ -Fano type with Mori fibration structures is bounded.*

1.1. Sketch of the proof. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$. If $\dim Z = 0$, then X is a \mathbb{Q} -factorial terminal Fano 3-fold with Picard number one, which is bounded by Kawamata [9]. So we only need to consider the case when $\dim Z > 0$.

We recall the following theorem from [7], by which we proved the birational boundedness of 3-folds of ϵ -Fano type.

Theorem 1.4 ([7, Proof of Corollaries 1.5, 1.8]). *Fix $0 < \epsilon < 1$. Then there exist positive integers N_ϵ and V_ϵ depending only on ϵ , with the following property:*

If X is a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$.

(1) If $\dim Z = 1$ (i.e. $Z = \mathbb{P}^1$), take a general fiber F of f , then

(1-1) $|-3K_X + N_\epsilon F|$ is ample and gives a birational map;

(1-2) $(-3K_X + N_\epsilon F)^3 \leq V_\epsilon$.

(2) If $\dim Z = 2$, then there exists a very ample divisor H on Z such that

*(2-1) $|-2K_X + N_\epsilon f^*H|$ is ample and gives a birational map;*

*(2-2) $(-2K_X + N_\epsilon f^*H)^3 \leq V_\epsilon$.*

Therefore, to show the boundedness, it suffices to show the boundedness of Gorenstein indices.

For convenience, we define G and \mathcal{F}_X as following.

Definition 1.5. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ such that $\dim Z > 0$. Keep the notation in Theorem 1.4.

(1) Define the projective smooth surface G to be a general fiber F (resp. a general element of $|f^*(H)|$) if $\dim Z = 1$ (resp. $\dim Z = 2$).

(2) Define the torsion free sheaf $\mathcal{F}_X := T_X^1 \oplus \mathcal{O}_X(N_\epsilon G)$.

Note that by Theorem 1.4 and the fact that G is nef, it is easy to see that $-K_X + N_\epsilon G$ is ample with $(-K_X + N_\epsilon G)^3 \leq V_\epsilon$. Also we remark that G is a del Pezzo surface (resp. conic bundle over a general H) if $\dim Z = 1$ (resp. $\dim Z = 2$).

Following the idea of Kollár–Miyaoka–Mori–Takagi [14], we can prove the pseudo-effectivity of $c_2(\mathcal{F}_X)$.

Theorem 1.6. *Fix $0 < \epsilon < 1$. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ such that $\dim Z > 0$. Keep the notation in Definition 1.5. Then $c_2(\mathcal{F}_X)$ is pseudo-effective.*

Following the idea of Kawamata [9], after bounding $(-K_X) \cdot c_2(X)$ from below, we can get an upper bound for Cartier index of K_X , which implies the desired boundedness.

Theorem 1.7. *Fix $0 < \epsilon < 1$. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$ such that $\dim Z > 0$, then*

- (1) $(-K_X) \cdot c_2(X) \geq -M_\epsilon$;
- (2) $r_\epsilon K_X$ is Cartier,

where $M_\epsilon = 5V_\epsilon + 12N_\epsilon$ and $r_\epsilon = (24 + M_\epsilon)!$, N_ϵ and V_ϵ are the numbers defined in Theorem 1.4.

2. PROOF OF THEOREMS

2.1. Notation and conventions. We adopt the standard notation and definitions in [10] and [15], and will freely use them.

A pair (X, B) consists of a normal projective variety X and an effective \mathbb{Q} -divisor B on X such that $K_X + B$ is \mathbb{Q} -Cartier.

The pair (X, B) is called a *log Fano pair* if $-(K_X + B)$ is ample.

Let $f : Y \rightarrow X$ be a log resolution of the pair (X, B) , write

$$K_Y = f^*(K_X + B) + \sum a_i F_i,$$

where $\{F_i\}$ are distinct prime divisors. For some $\epsilon \in [0, 1]$, the pair (X, B) is called

- (a) *ϵ -kawamata log terminal* (ϵ -klt, for short) if $a_i > -1 + \epsilon$ for all i ;
- (b) *ϵ -log canonical* (ϵ -lc, for short) if $a_i \geq -1 + \epsilon$ for all i ;
- (c) *terminal* if $a_i > 0$ for all f -exceptional divisors F_i and all f .

Usually we write X instead of $(X, 0)$ in the case $B = 0$. Note that 0-klt (resp. 0-lc) is just klt (resp. lc) in the usual sense.

In particular, a normal projective variety X is of ϵ -Fano type if there exists an effective \mathbb{Q} -divisor B such that (X, B) is an ϵ -klt log Fano pair.

A projective morphism $f : X \rightarrow Z$ between normal projective varieties is called a *Mori fibration* (or *Mori fiber space*) if

- (1) X is \mathbb{Q} -factorial with terminal singularities;
- (2) f is a *contraction*, i.e., $f_*\mathcal{O}_X = \mathcal{O}_Z$;
- (3) $-K_X$ is ample over Z ;
- (4) $\rho(X/Z) = 1$;
- (5) $\dim X > \dim Z$.

We say that X is with a *Mori fibration structure* if there exists a Mori fibration $X \rightarrow Z$. In particular, in this situation, X has at most \mathbb{Q} -factorial terminal singularities by definition.

A collection of varieties $\{X_t\}_{t \in T}$ is said to be *bounded* (resp. *rationally bounded*) if there exists $h : \mathcal{X} \rightarrow S$ a projective morphism between schemes of finite type such that each X_t is isomorphic (resp. birational) to \mathcal{X}_s for some $s \in S$.

2.2. Pseudo-effectivity of $c_2(\mathcal{F}_X)$. We recall a criterion of pseudo-effectivity of second Chern classes due to Miyaoka [16].

Definition 2.1 (cf. [16]). Let X be an n -dimensional normal projective variety. A torsion free sheaf \mathcal{E} is called *generically semi-positive* (or *generically nef*) if one of the following equivalent conditions holds:

- (1) For every quotient torsion free sheaf $\mathcal{E} \rightarrow \mathcal{L}$ and any ample divisors H_i ,

$$c_1(\mathcal{L}) \cdot H_1 \cdot H_2 \cdots H_{n-1} \geq 0.$$

- (2) $\mathcal{E}|_C$ is nef for a general curve $C = D_1 \cap \dots \cap D_{n-1}$ for general $D_i \in |m_i H_i|$ and $m_i \gg 0$ and any ample divisors H_i .

Theorem 2.2 ([16, Theorem 6.1]). *Let X be a normal projective variety which is smooth in codimension 2. Let \mathcal{E} be a torsion free sheaf on X such that*

- (1) $c_1(\mathcal{E})$ is a nef \mathbb{Q} -Cartier divisor, and
(2) \mathcal{E} is generically semi-positive.

Then $c_2(\mathcal{E})$ is pseudo-effective.

To check the generic semi-positivity of \mathcal{F}_X , it suffices to check that of T_X^1 , which is proved by the following theorem.

Theorem 2.3 ([14, Proof of 1.2 (1)]). *Let (X, B) be a \mathbb{Q} -factorial klt log Fano pair such that X is smooth in codimension 2. Then T_X^1 is generically semi-positive.*

This theorem is essentially implicated by [14, Proof of 1.2 (1)], combining a structure theorem for the closed cone of nef curves (cf. [4, Corollary 1.3.5]) and deformation theory of rational curves (cf. [12, (1.3) Corollary]).

Proof of Theorem 1.6. Recall that X is of Fano type and with \mathbb{Q} -factorial terminal singularities. Since T_X^1 is generically semi-positive by Theorem 2.3 and G is nef, $\mathcal{F}_X = T_X^1 \oplus \mathcal{O}_X(N_\epsilon G)$ is again generically semi-positive. Since $c_1(\mathcal{F}_X) = -K_X + N_\epsilon G$ is ample, $c_2(\mathcal{F}_X)$ is pseudo-effective by Theorem 2.2. \square

2.3. Upper bound of Gorenstein indices. In this subsection, we prove Theorem 1.7. We start from the estimate of $(-K_X) \cdot c_2(X)$.

Proof of Theorem 1.7(1). Note that

$$c_2(\mathcal{F}_X) = c_2(T_X^1 \oplus \mathcal{O}_X(N_\epsilon G)) = c_2(X) - K_X \cdot N_\epsilon G.$$

By Theorem 1.6, $c_2(\mathcal{F}_X)$ is pseudo-effective, and thus

$$(-K_X + N_\epsilon G) \cdot (c_2(X) - K_X \cdot N_\epsilon G) \geq 0.$$

Hence

$$(-K_X) \cdot c_2(X) \geq -(-K_X + N_\epsilon G) \cdot (-K_X) \cdot N_\epsilon G - N_\epsilon G \cdot c_2(X).$$

It suffices to prove the following lemma.

Lemma 2.4. *The following inequalities hold:*

- (1) $(-K_X + N_\epsilon G) \cdot (-K_X) \cdot N_\epsilon G \leq V_\epsilon;$
- (2) $G \cdot c_2(X) \leq 12 + 4V_\epsilon/N_\epsilon.$

Proof. Recall that $-K_X$ is big, G is nef, and $-K_X + N_\epsilon G$ is ample with $(-K_X + N_\epsilon G)^3 \leq V_\epsilon.$

For statement (1),

$$\begin{aligned} & (-K_X + N_\epsilon G) \cdot (-K_X) \cdot N_\epsilon G \\ & \leq (-K_X + N_\epsilon G) \cdot (-K_X + N_\epsilon G) \cdot N_\epsilon G \\ & \leq (-K_X + N_\epsilon G)^3 \leq V_\epsilon. \end{aligned}$$

Now we prove statement (2).

If $\dim Z = 1$, then G is a del Pezzo surface and $G \cdot c_2(X) = c_2(G) \leq 11.$

If $\dim Z = 2$, by the exact sequence

$$0 \rightarrow T_G^1 \rightarrow T_X^1|_G \rightarrow \mathcal{N}_{G/X} \rightarrow 0,$$

we have

$$\begin{aligned} & G \cdot c_2(X) \\ & = c_2(G) + c_1(G) \cdot c_1(\mathcal{N}_{G/X}) \\ & = 12\chi(\mathcal{O}_G) - K_G^2 - K_G \cdot G|_G. \end{aligned}$$

Note that G is a conic bundle over H , hence $\chi(\mathcal{O}_X) = 1 - g(H) \leq 1.$ On the other hand,

$$\begin{aligned} & -K_G^2 - K_G \cdot G|_G \\ & = -(K_X + G)^2 \cdot G - (K_X + G) \cdot G^2 \\ & = -K_X \cdot (K_X + 3G) \cdot G \\ & \leq -K_X \cdot (N_\epsilon + 3)G \cdot G \\ & = (-K_X + N_\epsilon G) \cdot (N_\epsilon + 3)G^2 \\ & \leq (-K_X + N_\epsilon G) \cdot \frac{N_\epsilon + 3}{N_\epsilon^2} (-K_X + N_\epsilon G)^2 \\ & \leq \frac{N_\epsilon + 3}{N_\epsilon^2} V_\epsilon \leq \frac{4V_\epsilon}{N_\epsilon}. \end{aligned}$$

Hence $G \cdot c_2(X) \leq 12 + 4V_\epsilon/N_\epsilon.$ □

By this lemma,

$$(-K_X) \cdot c_2(X) \geq -(5V_\epsilon + 12N_\epsilon).$$

Hence Theorem 1.7(1) is proved. □

By Reid's Riemann–Roch formula, we can get the upper bound of Gorenstein indices. This method highly depends on the classification of 3-dimensional terminal singularities.

Proof of Theorem 1.7(2). Recall that X has at most terminal singularities. By Reid's Riemann–Roch formula (cf. [8, Lemmas 2.2, 2.3] or [17, (10.3)]), we have

$$\chi(\mathcal{O}_X) = \frac{1}{24}(-K_X) \cdot c_2(X) + \frac{1}{24} \sum (r_i - \frac{1}{r_i}),$$

where r_i are indices of cyclic quotient terminal singularities obtained by deforming singularities of X locally. Note that $\chi(\mathcal{O}_X) = 1$ since X is of Fano type. Hence by Theorem 1.7(1),

$$\sum (r_i - \frac{1}{r_i}) \leq 24 + M_\epsilon.$$

In particular, $r_i \leq 24 + M_\epsilon$. Hence $(24 + M_\epsilon)!$ is divisible by $\text{l.c.m.}\{r_i\}$, which is just the Cartier index of K_X . \square

2.4. Proof of the main theorem.

Proof of Theorem 1.3. Let X be a 3-fold of ϵ -Fano type with a Mori fibration $f : X \rightarrow Z$. If $\dim Z = 0$, then X is a \mathbb{Q} -factorial terminal Fano 3-fold with Picard number one, which is bounded by Kawamata [9]. So we only need to consider the case when $\dim Z > 0$.

Keep the notation in Theorem 1.4. By Theorem 1.7, $L := r_\epsilon(-K_X + N_\epsilon G)$ is a Cartier ample divisor. Recall that X is of Fano type. By Kollár's effective base point free theorem (cf. [11, 1.1 Theorem, 1.2 Lemma]), $720L$ is base point free and $4321L$ is very ample. On the other hand, $L^3 \leq r_\epsilon^3 V_\epsilon$. Hence X is a subvariety of projective spaces with bounded degree. Such X forms a bounded family by the boundedness of Chow variety. \square

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